

COLD IDEAL EQUATION OF STATE FOR STRONGLY MAGNETIZED NEUTRON-STAR MATTER:
EFFECTS ON MUON PRODUCTION AND PION CONDENSATIONIN-SAENG SUH¹ AND G. J. MATHEWS²

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ABSTRACT

Neutron stars with very strong surface magnetic fields have been suggested as the site for the origin of observed soft gamma repeaters (SGRs). In this paper we investigate the influence of such strong magnetic fields on the properties and internal structure of these strongly magnetized neutron stars (magnetars). We study properties of a degenerate equilibrium ideal neutron-proton-electron (*npe*) gas with and without the effects of the anomalous nucleon magnetic moments in a strong magnetic field. The presence of a sufficiently strong magnetic field changes the ratio of protons to neutrons as well as the neutron drip density. We also study the appearance of muons as well as pion condensation in strong magnetic fields. We discuss the possibility that boson condensation in the interior of magnetars might be a source of SGRs.

Subject headings: stars: interiors — stars: magnetic fields — stars: neutron

1. INTRODUCTION

Among the more than two thousand observed cosmological gamma-ray bursts (GRBs), four recurrent sources, so-called soft gamma repeaters (SGRs), have been identified and a fifth has probably been observed (Hurley 2000). They are believed to be a new class of γ -ray transients separate from the source of classical GRBs. Observations of SGR 0526-66 (Mazets et al. 1979), SGR 1806-20 (Murakami et al. 1994), and SGR 1900+14 (Kouveliotou et al. 1998) with RXTE, ASCA, and BeppoSAX have confirmed the fact that these SGRs are newly born neutron stars that have very large surface magnetic fields (up to 10^{15} G) based upon measurements of the spin-down timescale. Recently, SGR 1627-41 has also been discovered by BATSE (Woods et al. 1999). It is estimated that its magnetic field could be $B \gtrsim 5 \times 10^{14}$ G. The most recent source is SGR 1801-23 (Cline et al. 2000) observed by *Ulysses*, BATSE, and KONUS-Wind. Such stars have been named magnetars (Duncan & Thompson 1992, Thompson & Duncan 1995). [Note however that recently Harding et al. (1999) and Marsden et al. (1999) have suggested that if relativistic wind outflow continuously dominates the spin-down of SGR 1806-20 and SGR 1900+14, then the surface dipole field may be too low to be consistent with a magnetar model.]

Magnetars have also been suggested as the site for anomalous X-ray pulsars (AXPs) (van Paradijs, Taam, & van den Heuvel 1995) such as 1E 1841-045 (Kes 73) (Gotthelf, Vasisht, & Dotani 1999), RX J0720.4-3125 (Haberl et al. 1997), and 1E 2259+586 (Rho & Petre 1997). However, as another scenario, Chatterjee et al. (1999) proposed that these AXPs should have magnetic fields similar to neutron stars inferred for ordinary radio pulsars and binary X-ray pulsars in an accretion-powered X-ray emission model.

Whether or not magnetars are the source of SGRs or AXPs, as relics of stellar interiors, the study of the magnetic fields in and around degenerate stars should give important information on the role such fields play in star formation and stellar evolution. Indeed, the origin and evolution of stellar magnetic fields remains obscure. As early as Ginzburg (1964) and Woltjer (1964) it was proposed that the magnetic flux ($\Phi_B \sim BR^2$) of a star is conserved during its evolution and subsequent col-

lapse to form a remnant white dwarf or neutron star. A main sequence star with radius $R \sim 10^{11}$ cm and surface magnetic field $B \sim 10-10^4$ G [magnetic A-type stars have typical surface fields $\lesssim 10^4$ G (Shapiro & Teukolsky 1983)] would thus collapse to form a white dwarf with $R \sim 10^9$ cm and $B \sim 10^5-10^8$ G, or a neutron star with $R \sim 10^6$ cm and $B \sim 10^{11}-10^{14}$ G. Indeed, shortly after their discovery (Hewish et al. 1968) pulsars were identified as rotating neutron stars (Gold 1968) with magnetic fields $B \sim 10^{11}-10^{13}$ G consistent with magnetic field amplification by flux conservation.

Recently, Thompson and Duncan (1993) have invoked a convective dynamo mechanism to suggest that the magnetic dipole field of young neutron stars could realistically reach values of the order of $10^{14}-10^{15}$ G, i.e., 10^2-10^3 times stronger than ordinary pulsars. Moreover, the internal magnetic field of a star may not necessarily be reflected in its surface magnetic field (Ruderman 1980, Galloway, Proctor, & Weiss 1977). Therefore, the total strength of internal magnetic fields remains unknown. Nevertheless, it is expected that appreciably higher magnetic fields can exist in the interiors of neutron stars (Ruderman 1980).

Ultimately, the allowed internal field strength of a star is constrained by the scalar virial theorem (cf. Chandrasekhar & Fermi 1953, Shapiro & Teukolsky 1983),

$$2T + W + 3\Pi + \mathcal{M} = 0,$$

where T is the total kinetic energy, W is the gravitational potential energy, Π is the internal energy, and \mathcal{M} is the magnetic energy. For a star of size R and mass M , this gives a maximum interior field strength of $B \sim 2 \times 10^8 (M/M_\odot)(R/R_\odot)^{-2}$ G. For neutron stars with $R \approx 10$ km and $M \approx M_\odot$, the maximum interior field strength could thus reach $B \lesssim 10^{18}$ G (Lerche & Schramm 1977). Numerical studies (Bocquet et al. 1995) have confirmed that neutron stars with ultrastrong magnetic fields are stable up to the order of 10^{18} G. They also have found that for such values the maximum mass of neutron stars increases by 13 - 29 % relative to the maximum mass of non-magnetized neutron stars. This is similar to the case of magnetic white dwarfs (Suh & Mathews 2000a).

The strength of the internal magnetic field in a neutron star

could, in principle, be constrained by any observable consequences of a strong magnetic field. For example, rapid motion of neutron stars may be due to anisotropic neutrino emission induced by a strong magnetic field (e.g., see Janka 1998). One could also consider the effect of magnetic fields on the thermal evolution (Heyl & Hernquist 1997, Baiko & Yakovlev 1999) and the maximum mass (Vshivtsev & Serebryakova 1994) of neutron stars. Recently, Chakrabarty et al. (1995) have investigated the gross properties of cold nuclear matter in a strong magnetic field in the context of a relativistic Hartree model and have applied their equation of state to obtain the maximum masses and radii for magnetic neutron stars.

Since strong interior magnetic fields modify the nuclear equation of state for degenerate stars, their mass-radius relation will be changed relative to that of nonmagnetic stars. Recently, we have obtained a revised mass-radius relation for magnetic white dwarfs (Suh & Mathews 2000a) with the equation of state for electrons in a strong magnetic field. For strong internal magnetic fields of $B \sim 4.4 \times (10^{11} - 10^{13})$ G, we have found that both the mass and radius increase distinguishably and the mass-radius relation of some observed magnetic white dwarfs may be better fit if strong internal fields are assumed.

In this work, we now extend this study to an investigation of the effect of magnetic fields on the internal properties of neutron stars as well. If ultrastrong magnetic fields exist in the interior of neutron stars, such fields will primarily affect the behavior of the residual charged particles. Moreover, contributions from the anomalous magnetic moment (AMM) of the particles in a strong magnetic field should also be significant (Broderick, Prakash, & Lattimer 2000). In particular, in a strong magnetic field, complete spin polarization of the neutrons occurs due to the interaction of the neutron magnetic moment with the magnetic field. Therefore, we consider both cases with and without the effects of the AMM.

Even so, standard internal properties such as the nuclear equation of state, neutron drip, and the threshold density of new particles will be modified by a strong magnetic field. For purposes of illustration, we will consider a degenerate ideal non-interacting neutron-proton-electron (*npe*) gas in equilibrium (Shapiro & Teukolsky 1983). We find that under conditions of charge neutrality and chemical equilibrium, the presence of a sufficiently strong magnetic field changes the ratio of protons to neutrons as well as the threshold density for the appearance of muons and pion condensation.

In Sec. 2 we first review the properties of electrons and describe the equation of state for a particle gas in an external magnetic field. In Sec. 3 we first consider the case without the particle AMM and we derive the proton to neutron ratio in an ideal *npe* gas for the lowest Landau level analytically. We also numerically obtain the neutron appearance density and proton concentration, Y_p , in magnetic fields. In Sec. 4 we include an effect of the particle AMM on the equation of state and the adiabatic index. In Sec. 5 we calculate the muon threshold density in a magnetic field. We consider pion production and condensation in strongly magnetized neutron-star matter in Sec. 6. Finally, we discuss the possibility that boson condensation in the interior of magnetars is a source of SGRs.

2. EQUATION OF STATE FOR PARTICLES IN A MAGNETIC FIELD

2.1. Electrons and muons

Let us begin by considering the properties of an electron in an external magnetic field (Landau & Lifshitz 1938, Johnson &

Lippmann 1949, Canuto & Chiu 1968, Schwinger 1988). For a sufficiently high field, the energy states of an electron in a magnetic field are quantized and its properties are modified accordingly. The cyclotron energy becomes comparable to the electron rest-mass energy and the electrons in the excited Landau levels become relativistic.

In order to investigate these effects for electrons we must first solve the Dirac equation in an external, static, and homogeneous magnetic field. With a uniform magnetic field B along the z -axis and the convenient choice of gauge for the vector potential, *i.e.*, $A_0 = 0$ and $\vec{A} = (-yB, 0, 0)$, we obtain the electron wavefunctions and energy dispersion in a magnetic field (Johnson & Lippmann 1949, Schwinger 1988). The dispersion relation relation for an electron propagating through a magnetic field is

$$E_{n_f^e}^e = [p_z^e c^2 + m_e^2 c^4 + 2\hbar c e B n_f^e]^{1/2} + m_e c^2 \kappa_e, \quad (1)$$

where $n_f^e = n + \frac{1}{2} + s_z^e$, in which n is the principal quantum number of the Landau level, $s_z^e = \pm 1/2$ is the electron spin projection onto the magnetic field direction, e is the electron charge, c is the speed of light, \hbar is Planck's constant, p_z^e is the electron momentum along the z -axis, and m_e is the rest mass of the electron. Here, let us use a definition $\gamma_e \equiv B/B_c^e$ where $B_c^e = m_e^2 c^3 / e \hbar = 4.414 \times 10^{13}$ G. In Eq. (1), $\kappa_e = -(\alpha/4\pi)\gamma_e$, for $\gamma_e \ll 1$ and $\kappa_e = (\alpha/4\pi)[\ln(2\gamma_e) - (C + \frac{3}{2})]^2 + \dots$, for $\gamma_e \gg 1$, where $\alpha \simeq 1/137$ and $C = 0.577$ is Euler's constant (Schwinger 1988).

The main modification of an electron in a magnetic field comes from the available density of states for the electrons (Landau & Lifshitz 1938). The electron state density in the absence of a magnetic field,

$$\frac{2}{\hbar^3} \int \frac{d^3 \vec{p}}{(2\pi)^3}, \quad (2)$$

is replaced with

$$\frac{2}{\hbar^2 c} \sum_n \sum_{s_z^e} \int \frac{eB}{(2\pi)^2} dp_z \quad (3)$$

in a magnetic field. This modification affects the thermodynamic properties of the electron gas.

Let us consider a gas of electrons at zero temperature in a magnetic field (Blandford & Hernquist 1982). From Eq. (1) we can define the electron Fermi energy E_F^e for an arbitrary Landau level n_f^e as

$$E_F^e \equiv [m_e^2 c^4 + p_F^e c^2 + 2\hbar c e B n_f^e]^{1/2} + m_e c^2 \kappa_e. \quad (4)$$

Here p_F^e denotes the electron Fermi momentum.

Now we can obtain all of thermodynamic quantities in terms of the Fermi energy, Eq. (4), and the phase space integration, Eq. (3), in a magnetic field. The number density of electrons in a magnetic field is then given by

$$n_e = \frac{\gamma_e}{2\pi^2} \left(\frac{m_e c}{\hbar} \right)^3 \zeta_e(\epsilon_e), \quad (5)$$

where ϵ_e is defined as $\epsilon_e \equiv E_F^e / m_e c^2$ and

$$\zeta_e(\epsilon_e) = \sum_{n=0}^{n_{\max}^e} \sum_{s_z^e} \sqrt{(\epsilon_e - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)}. \quad (6)$$

The maximum Landau level n_{max}^e for a given electron Fermi energy ϵ_e and magnetic field strength γ_e is given by

$$n_{max}^e = \text{int} \left[\frac{(\epsilon_e - \kappa_e)^2 - 1}{2\gamma_e} - \left(\frac{1}{2} + s_z^e \right) \right] \geq n, \quad (7)$$

where $\text{int}[x]$ means an integer value of the argument x . The pressure of an ideal electron gas in a magnetic field is then

$$P_e = \frac{\gamma_e}{4\pi^2} m_e c^2 \left(\frac{m_e c}{\hbar} \right)^3 \Phi_e(\epsilon_e), \quad (8)$$

where

$$\Phi_e(\epsilon_e) = \sum_{n=0}^{n_{max}^e} \sum_{s_z^e} \left[(\epsilon_e - \kappa_e) \sqrt{(\epsilon_e - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)} \right. \\ \left. - (1 + 2\gamma_e n_f^e) \ln \left(\frac{(\epsilon_e - \kappa_e) + \sqrt{(\epsilon_e - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)}}{\sqrt{1 + 2\gamma_e n_f^e}} \right) \right]. \quad (9)$$

Similarly, the energy density is

$$\mathcal{E}_e(\epsilon_e) = \frac{\gamma_e}{4\pi^2} m_e c^2 \left(\frac{m_e c}{\hbar} \right)^3 \chi_e(\epsilon_e), \quad (10)$$

where

$$\chi_e(\epsilon_e) = \frac{1}{2} \sum_{n=0}^{n_{max}^e} \sum_{s_z^e} \left[(\epsilon_e - \kappa_e) \sqrt{(\epsilon_e - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)} \right. \\ \left. + (1 + 2\gamma_e n_f^e) \ln \left(\frac{(\epsilon_e - \kappa_e) + \sqrt{(\epsilon_e - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)}}{\sqrt{1 + 2\gamma_e n_f^e}} \right) \right]. \quad (11)$$

From these, we obtain the energy per electron

$$E_e(\epsilon_e) = m_e c^2 \frac{\chi_e(\epsilon_e)}{\zeta_e(\epsilon_e)}. \quad (12)$$

Note, that as γ_e goes to zero, Eqs. (6) and (8) - (12) recover exactly the usual non-magnetic equation of state for electrons (Suh & Mathews 2000a).

We can obtain similar quantities for muons simply by replacing the electron quantities by the corresponding muon quantities (e.g., replace $\gamma_e = B/B_c^e$ by $\gamma_\mu = B/B_c^\mu$, where $B_c^\mu = m_\mu^2 c^3 / e\hbar$). The muon AMM κ_μ has nearly the same value as κ_e . The difference is only $\kappa_\mu - \kappa_e \simeq 0.59 \times 10^{-5}$ (Schwinger 1973) [The experimental value for this difference is 0.63×10^{-5} (Grandy 1991)].

2.2. Protons

Although the proton mass is much greater than the electron mass, magnetic effects on protons can be as important as those on electrons (Lai & Shapiro 1991). For instance, the proton pressure is always much smaller than the electron pressure at low density. But, ignoring the influence of the magnetic field on protons would lead to the unphysical result of proton pressure dominance at low density. Therefore, whenever the magnetic field significantly affects the electrons, it also affects the protons.

The energy dispersion relation for protons $E_{n_f}^p$ for an arbitrary Landau level in a magnetic field is:

$$E_{n_f}^p = \left[p_z^p c^2 + m_p^2 c^4 \left[\left\{ 1 + 2\gamma_p n_f^p \right\}^{1/2} - s_z^p \frac{\mu_N^p B}{m_p c^2} \right]^2 \right]^{1/2}, \quad (13)$$

where $n_f^p = n + \frac{1}{2} - s_z^p$, n is the principal quantum number of the Landau level, $s_z^p = \pm \frac{1}{2}$ is the z component of the proton spin, p_z^p is the proton momentum along the z -axis, and $\mu_N^p = (e\hbar/m_p c)\kappa_p$, with $\kappa_p = 2.79$, is the proton anomalous magnetic moment.

The proton number density in a magnetic field is :

$$n_p = \frac{\gamma_p}{2\pi^2} \left(\frac{m_p c}{\hbar} \right)^3 \zeta_p(\epsilon_p), \quad (14)$$

where $\epsilon_p \equiv E_F^p / m_p c^2$ and

$$\zeta_p(\epsilon_p) = \sum_{n=0}^{n_{max}^p} \sum_{s_z^p} \sqrt{\epsilon_p^2 - \tilde{m}_p^2}, \quad (15)$$

$$\tilde{m}_p = \sqrt{1 + 2\gamma_p n_f^p} - s_z^p \kappa_p \gamma_p. \quad (16)$$

The maximum Landau level n_{max}^p for a proton in a magnetic field is given by

$$n_{max}^p = \text{int} \left[\frac{(\epsilon_p + s_z^p \kappa_p \gamma_p)^2 - 1}{2\gamma_p} - \left(\frac{1}{2} - s_z^p \right) \right]. \quad (17)$$

The pressure of a proton gas in a magnetic field is

$$P_p = \frac{\gamma_p}{4\pi^2} m_p c^2 \left(\frac{m_p c}{\hbar} \right)^3 \Phi_p(\epsilon_p), \quad (18)$$

where

$$\Phi_p(\epsilon_p) = \sum_{n=0}^{n_{max}^p} \sum_{s_z^p} \left[\epsilon_p \sqrt{\epsilon_p^2 - \tilde{m}_p^2} - \tilde{m}_p^2 \ln \left(\frac{\epsilon_p + \sqrt{\epsilon_p^2 - \tilde{m}_p^2}}{\sqrt{\tilde{m}_p}} \right) \right]. \quad (19)$$

The energy density is

$$\mathcal{E}_p(\epsilon_p) = \frac{\gamma_p}{4\pi^2} m_p c^2 \left(\frac{m_p c}{\hbar} \right)^3 \chi_p(\epsilon_p), \quad (20)$$

where

$$\chi_p(\epsilon_p) = \frac{1}{2} \sum_{n=0}^{n_{max}^p} \sum_{s_z^p} \left[\epsilon_p \sqrt{\epsilon_p^2 - \tilde{m}_p^2} + \tilde{m}_p^2 \ln \left(\frac{\epsilon_p + \sqrt{\epsilon_p^2 - \tilde{m}_p^2}}{\sqrt{\tilde{m}_p}} \right) \right]. \quad (21)$$

2.3. Neutrons

A neutral Dirac fermion can interact with an external electromagnetic field by means of the Pauli non-minimal coupling. Then the energy dispersion relation E_F^n for neutrons in a magnetic field is given by

$$E_F^n = \left[p_\parallel^2 c^2 + (\sqrt{m_n^2 c^4 + p_\perp^2 c^2} + s_z^n \mu_N^n B)^2 \right]^{1/2} \quad (22)$$

where $s_z^n = \pm \frac{1}{2}$ is the neutron spin projection onto the magnetic field direction, p_\parallel and p_\perp are the components of the neutron

momentum parallel and perpendicular to the magnetic field, and $\mu_N^n = (e\hbar/m_n c)\kappa_n$, with $\kappa_n = -1.91$, is the neutron anomalous magnetic moment.

In the calculations of thermodynamic quantities, the phase-space integration separates into two steps which involve integration over p_{\parallel} and p_{\perp} (e.g., see Broderick, Prakash, & Lattimer 2000). Then the neutron number density in a magnetic field is given by

$$n_n = \frac{1}{2\pi^2} \left(\frac{m_n c}{\hbar} \right)^3 \zeta_n(\epsilon_n), \quad (23)$$

where

$$\begin{aligned} \zeta_n(\epsilon_n) = & \sum_{s_z^n} \frac{1}{3} \left[[\epsilon_n^2 - \tilde{m}_n^2]^{3/2} \right. \\ & \left. + \frac{3}{2} s_z^n \kappa_n \gamma_n \left[\tilde{m}_n \sqrt{\epsilon_n^2 - \tilde{m}_n^2} + \epsilon_n^2 \left\{ \arcsin \left(\frac{\tilde{m}_n}{\epsilon_n} \right) - \frac{\pi}{2} \right\} \right] \right] \end{aligned} \quad (24)$$

In the above Eqs (23) and (24), we have defined $\epsilon_n \equiv E_F^n/m_n c^2$, $\gamma_n \equiv B/B_c^n$, $B_c^n = e\hbar/m_n^2 c^3$, and

$$\tilde{m}_n = 1 + s_z^n \kappa_n \gamma_n. \quad (25)$$

The pressure of a neutron gas in a magnetic field is

$$P_n = \frac{1}{24\pi^2} m_n c^2 \left(\frac{m_n c}{\hbar} \right)^3 \Phi_n(\epsilon_n), \quad (26)$$

where

$$\Phi_n(\epsilon_n) = \sum_{s_z^n} \frac{1}{2} \left[\Phi_n^0 + s_z^n \kappa_n \gamma_n \Phi_n^{\kappa} \right]. \quad (27)$$

In equation (27),

$$\Phi_n^0 = (2\epsilon_n^2 - 4 - \tilde{m}_n^2 + 9\kappa_n^2 \gamma_n^2) \mathcal{P} + (4 - \tilde{m}_n^2 - 9\kappa_n^2 \gamma_n^2) \tilde{m}_n^2 \mathcal{Q}, \quad (28)$$

$$\begin{aligned} \Phi_n^{\kappa} = & \frac{1}{3} \tilde{m}_n \left[2\mathcal{P} + 4\{\tilde{m}_n^2 + \tilde{m}_n + 6(\kappa_n^2 \gamma_n^2 - 1)\} \mathcal{Q} \right] \\ & + \left[\frac{10}{3} \epsilon_n^2 + 8(\kappa_n^2 \gamma_n^2 - 1) \right] \mathcal{R}, \end{aligned} \quad (29)$$

where we defined the quantities:

$$\begin{aligned} \mathcal{P} &= \epsilon_n \sqrt{\epsilon_n^2 - \tilde{m}_n^2}, \quad \mathcal{Q} = \ln \left[\frac{\epsilon_n + \sqrt{\epsilon_n^2 - \tilde{m}_n^2}}{\tilde{m}_n} \right], \\ \mathcal{R} &= \epsilon_n \left(\arcsin \left(\frac{\tilde{m}_n}{\epsilon_n} \right) - \frac{\pi}{2} \right). \end{aligned} \quad (30)$$

Finally the energy density of a neutron gas in a magnetic field is

$$\mathcal{E}_n(\epsilon_n) = \frac{1}{8\pi^2} m_n c^2 \left(\frac{m_n c}{\hbar} \right)^3 \chi_n(\epsilon_n), \quad (31)$$

where

$$\chi_n(\epsilon_n) = \sum_{s_z^n} \frac{1}{6} \left[\chi_n^0 + s_z^n \kappa_n \gamma_n \chi_n^{\kappa} \right], \quad (32)$$

and

$$\chi_n^0 = 3[(2\epsilon_n^2 - \tilde{m}_n^2)\mathcal{P} - \tilde{m}_n^4 \mathcal{Q}], \quad \chi_n^{\kappa} = 4[\tilde{m}_n \mathcal{P} + \tilde{m}_n^3 \mathcal{Q} + 2\epsilon_n^2 \mathcal{R}]. \quad (33)$$

3. INVERSE β -DECAY AND NEUTRON APPEARANCE IN A STRONG MAGNETIC FIELD: WITHOUT THE ANOMALOUS MAGNETIC MOMENTS

Let us first consider the physics of an npe gas in a strong magnetic field without the anomalous magnetic moments of particles, i.e., $\kappa_j = 0$ ($j = e, p, n$). This illustrates the dominant physics at moderate magnetic field strength. For illustration, consider a homogeneous gas of free neutrons, protons, and electrons in β -equilibrium in a uniform magnetic field. At low densities, the most energetically favorable nucleus is ^{56}Fe which is the endpoint of thermonuclear reactions. As the density increases above $\sim 10^4 \text{ g cm}^{-3}$, electrons become unbound and relativistic. At sufficiently high densities, $\rho \gtrsim 8 \times 10^6 \text{ g cm}^{-3}$, protons in nuclei are converted into neutrons via inverse β -decay:



Since the neutrinos can escape, energy is transported away from the system. Thus, the equation of state in the star will be modified mainly due to the inverse β -decay. The reaction (34) can proceed whenever the electron acquires enough energy to exceed the mass difference between protons and neutrons, $Q = m_n - m_p = 1.293 \text{ MeV}$. The transformation of protons into neutrons, reaction (34), is effective whenever the β -decay reaction:



is slower than the rate of electron capture by protons. Reaction (35) is blocked if the density is high enough that all energetically available electron energy levels in the Fermi sea are occupied. Thus, there is a critical density for the onset of reaction (34).

Similar to the field-free case, we can take into account the above processes in an intense magnetic field. Assuming that a mixture of free neutrons, protons, and electrons are in equilibrium, then reaction (34) implies

$$\mu_e + \mu_p = \mu_n, \quad (36)$$

where $\mu_j \equiv E_F^j$ ($j = e, p, n, \mu, \pi$) is the chemical potential of the j th particle. We have set the neutrino chemical potential μ_{ν_e} to zero. Let us now define

$$x_j \equiv \frac{p_j^j}{m_j c}, \quad \epsilon_j \equiv \frac{\mu_j}{m_j c^2}, \quad \text{and} \quad \lambda_j \equiv \frac{\hbar}{m_j c}, \quad (37)$$

where λ_j is the Compton wavelength of the j th particle.

From chemical equilibrium, Eq. (36), we have

$$m_e c^2 \epsilon_e + m_p c^2 \epsilon_p = m_n c^2 \epsilon_n, \quad (38)$$

and charge neutrality gives

$$n_e = n_p. \quad (39)$$

In order to determine the equilibrium composition and hence the equation of state, the above equations should be solved simultaneously.

Consider now the minimum density at which neutrons first appear in a strong magnetic field. This neutron appearance density is determined by setting $n_n = 0$, or $\epsilon_n = 1 = (1 + x_n^2)^{1/2}$.

Since the protons at this density are nonrelativistic, i.e., $\epsilon_p^* \approx 1$, we approximately obtain $\epsilon_e^* \simeq 2.53$ as the specific electron chemical potential at which neutrons first appear according to Eq. (38) (Hereafter an asterisk is used to denote a threshold value for the appearance of new particles.) Therefore, we have $n_{max}^e = 0$ for electrons if $\gamma_e > (\epsilon_e^{*2} - 1)/2 \simeq 2.7$. That is, for $B > 2.7B_c^e \simeq 1.2 \times 10^{14}$ G, electrons reside in the lowest Landau level. Substantially n_{max}^p may not be zero for $2\gamma_p \ll 1$. But, in order to compare the two cases of higher-Landau-level and lowest-Landau-level occupation for the charged particles, we simply take $n_{max}^p = 0$ for protons.

Since Q and m_e are both much less than m_n , from Eqs. (38) and (39) we then obtain the proton-to-neutron ratio analytically when we assume that electrons and protons are in the lowest Landau level. Then

$$\frac{n_p}{n_n} = \left[\frac{(2m_n Q + C_n^2 n_n^{2/3})^2 - 4m_n^2 m_e^2}{4C_p^2 n_n^2 (m_n^2 + C_n^2 n_n^{2/3})} \right]^{1/2}, \quad (40)$$

where $C_p = 2\pi^2 m_p \lambda_p^3 / \gamma_p$, and $C_n^3 = 3\pi^2 m_n^3 \lambda_n^3$.

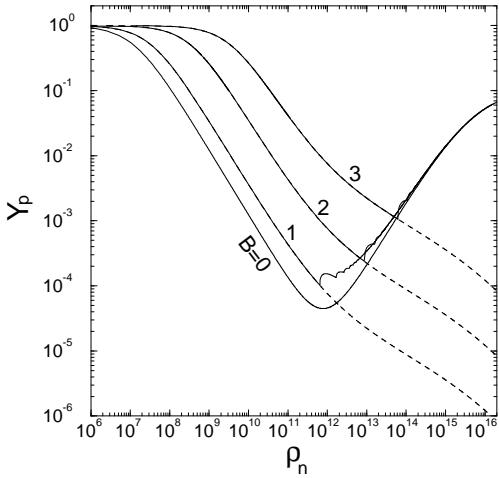


FIG. 1 The proton fraction $Y_p = n_p/n_B$ without the AMM as a function of the neutron density ρ_n (in unit g cm^{-3}) for the given value of $\log \gamma_e$, $\gamma_e = B/B_c^e$'s. The $B = 0$ line is the non-magnetic case. The dashed lines occur if charged particles are restricted in the lowest Landau level.

Figure 1 shows the proton fraction $Y_p = n_p/n_B$ (where $n_B = n_n + n_p$) as a function of the neutron density ρ_n for values of γ_e less than 10^3 . (For higher fields, we should take into account the nucleon AMM). In the nonmagnetic case, the conditions of charge neutrality and chemical equilibrium in a npe gas lead to a threshold for an increase in the proton concentration up to a value of $Y_p \simeq 1/9$ as ρ_n exceeds $\approx 10^{12} \text{ g cm}^{-3}$. This means that the inverse β decay is strongly suppressed by Pauli blocking in neutron-rich nuclear matter which consists only of a npe gas. However, in the case of a strong magnetic field, if we assume that electrons and protons are always in the lowest Landau level, then an increase in the concentration of protons does not occur even as ρ_n exceeds $10^{12} \text{ g cm}^{-3}$. That is, inverse β decay is not suppressed in magnetic fields. Far from suppressing the inverse β decay, the magnetic field instead catalyzes the reaction. This means that rapid neutron-star cooling can occur in a

strong magnetic field through the direct URCA process (Baiko & Yakovlev 1999, Leinson & Perez 1998). [Note that Baiko & Yakovlev (1999) have considered the direct URCA process at the core of a neutron star ($\rho > 10^{14} \text{ g cm}^{-3}$) and not its crust. In reality, the direct URCA process can never proceed at such low density as $10^{12} \text{ g cm}^{-3}$ because most protons are confined within the nuclei at $\rho < 10^{14} \text{ g cm}^{-3}$ in realistic neutron-star matter].

However, electrons and protons, actually, are not in the lowest Landau level for higher densities. Above a critical density, higher Landau levels begin to contribute to the chemical potential of the electrons and protons and hence particle number densities. Ultimately, discrete Landau levels become continuous and the proton concentration Y_p reverts to the nonmagnetic limit as the density increases. As a result, inverse β decay can still be suppressed at high densities in strong magnetic fields. Therefore, neutron-star rapid cooling may not be affected by the direct URCA process even though it is enhanced in strong magnetic fields. However, in order to enhance the cooling by the direct URCA process, one can invoke other mechanisms such as boson condensation (Tsuruta 1998), nucleon superfluidity (Yakovlev et al 1999), etc., if they exist.

The proton-to-neutron ratio, Eq. (40), gives the number density at which neutrons first appear:

$$n_n^*(B) = n_p^*(B) = n_e^*(B) = \frac{\gamma_e}{2\pi^2 \lambda_e^3} \left[\frac{Q^2 - m_e^2}{m_e^2} \right]^{1/2}. \quad (41)$$

Comparing with the zero-field result

$$n_n^*(0) = \frac{1}{3\pi^2} \frac{1}{\lambda_e^3} \left[\frac{Q^2 - m_e^2}{m_e^2} \right]^{3/2}, \quad (42)$$

we obtain the relative density at which neutrons appear in a strongly magnetized neutron star for $B > 2.7B_c^e$ (e.g., see Lai & Shapiro 1991):

$$\frac{\rho_n^*(B)}{\rho_n^*(0)} = \frac{3}{2} \gamma_e \left[\frac{m_e^2}{Q^2 - m_e^2} \right] = 0.277 \gamma_e. \quad (43)$$

We can see that the neutron appearance density increases linearly with the magnetic field B . This result is equivalent to one directly calculated from the general form (Shapiro & Teukolsky 1983),

$$\begin{aligned} \rho_n^*(B) &\simeq m_p n_e^*(B) \\ &= m_p \frac{\gamma_e}{2\pi^2 \lambda_e^3} \sum_{n=0}^{n_{max}^e} \sum_{s_z^e} \sqrt{(\epsilon_e^* - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)} \end{aligned} \quad (44)$$

with $\epsilon_e^* \simeq Q/m_e$.

4. EFFECTS OF ANOMALOUS MAGNETIC MOMENTS

Since $n_n = 0$ at neutron appearance, we have $\epsilon_n^* = 1 + s_z^n \kappa_n \gamma_n$. Then, from chemical equilibrium and charge neutrality, we obtain the electron Fermi energy at neutron appearance $\tilde{\epsilon}_e^*$ when the nucleon AMM is included:

$$\tilde{\epsilon}_e^* = \frac{\eta^2 - m_p^2 + m_e^2}{2m_e \eta}, \quad (45)$$

where $\eta = m_n (1 + s_z^n \kappa_n \gamma_n) + m_p s_z^p \kappa_p \gamma_p$. Note that η depends on both s_z^n and s_z^p . Of course Eq. (45) goes to $\epsilon_e^* \simeq Q/m_e$ when

$\kappa_{p,n} = 0$. In Eq. (45) η is similar to the neutron effective mass in a magnetic field because η becomes m_n when $\kappa_{p,n} = 0$. Hence, for $s_z^p = -1/2$, it is possible for η to have a negative value as γ_e increases. But this case is unphysical because $\tilde{\epsilon}_e^*$ becomes negative. Therefore, we take the proton spin $s_z^p = +1/2$ in this work. Now we can obtain the neutron appearance density when the nucleon AMM is included

$$\frac{\rho_n^*(B)}{\rho_n^*(0)} = \frac{3}{2} \frac{\gamma_e}{(\epsilon_e^* - 1)^{3/2}} \sum_{n=0}^{n_{max}^e} \sum_{s_z^e} \sqrt{(\tilde{\epsilon}_e^* - \kappa_e)^2 - (1 + 2\gamma_e n_f^e)}. \quad (46)$$

Here, n_{max}^e denotes the maximum electron Landau level at neutron appearance. The neutron threshold density $\rho_n^*(B)$ is plotted in figure 2 as a function of $\gamma_e = B/B_c^e$. We find that ρ_n^* is significantly affected by the nucleon AMM above $\log \gamma_e \approx 3$. Also, above this field strength, the neutron appearance density is split according to the orientation of the neutron spin.

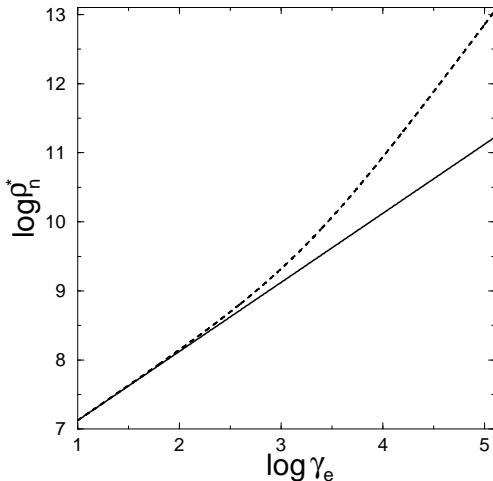


FIG. 2 Neutron appearance density ρ_n^* (in g cm^{-3}) as a function of γ_e . The solid line is the case without the AMM. The dashed line corresponds to $s_z^n = -1/2$ when the nucleon AMM's are included.

Now the equation of state for a npe gas at $\rho > \rho_n^*(B)$ can be determined in terms of the parameter ϵ_e . The charge neutrality condition, Eq. (33), at a fixed γ_e ,

$$\sum_{n=0}^{n_{max}^e} \sum_{s_z^e} p_F^e(\epsilon_e, s_z^e) = \sum_{n=0}^{n_{max}^p} \sum_{s_z^p} p_F^p(\epsilon_p, s_z^p) \quad (47)$$

gives ϵ_p for a given ϵ_e . We also obtain ϵ_n from the chemical equilibrium, Eq. (32). Note that both sides of Eq. (47) have the same statistical weight for the excited Landau level when the AMM of the proton is ignored. However, the statistical weight of protons is changed when the proton AMM is included.

Finally, we obtain the total baryon density $n_B = n_p + n_n$ and the proton concentration $Y_p = n_p/n_B$. In figure 3 we can see that the nucleon AMM above a field strength of $\gamma_e \gtrsim 10^3$ significantly affects the value of Y_p . The difference between the AMM and non-AMM results above a density of $\gtrsim 10^{12} \text{ g cm}^{-3}$ comes from the polarization of the proton spin. Besides, the equation of state, the mass-energy density $\rho = (\mathcal{E}_e + \mathcal{E}_p + \mathcal{E}_n)/c^2$ and the

pressure $P = P_e + P_p + P_n$, are straightforwardly determined. In calculations of the equation of state and the adiabatic index, we take $s_z^n = -1/2$.

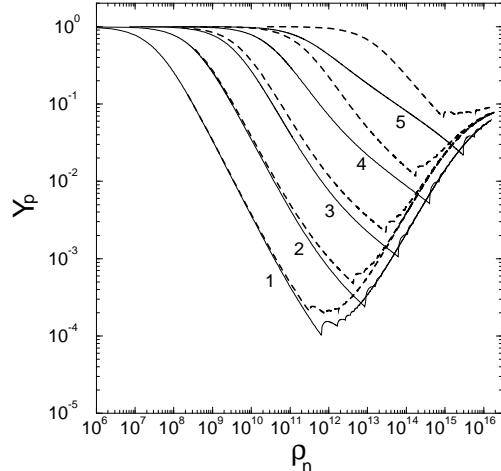


FIG. 3 The proton fraction $Y_p = n_p/n_B$ with the AMM included as a function of the neutron density ρ_n (in unit g cm^{-3}) for the given value of $\log \gamma_e$. The solid line is for the case without the nucleon AMM. The dashed lines are the cases of $s_z^n = -1/2$.

Lai & Shapiro (1991) have shown the equation of state and the adiabatic index for a field strength less than $\log \gamma_e = 2$. In this work, we calculate those for a field strength less than $\log \gamma_e = 5$. Figure 4 shows the equation of state for a strongly magnetized npe gas. At low density ($\rho < \rho_n^*$) and high magnetic field ($B \geq 4.4 \times 10^{14} \text{ G}$), we utilize the fact that electrons are nonrelativistic and are in the lowest Landau level as in Lai & Shapiro (1991). We plot the adiabatic index $\Gamma \equiv d \ln P / d \ln \rho$ as a function of ρ in Figure 5. We find a non-zero Γ at the neutron appearance as the magnetic field increases.

The adiabatic index is a crucial factor for understanding the global radial stability of a star as well as the local sound speed (Shapiro & Teukolsky 1983). In figure 5 we know that at neutron appearance Γ has a non-zero value and the minimum value of Γ increases as the magnetic field strength increases. But the nucleon AMM reduces the minimum of Γ at neutron appearance. A strange feature in figure 5 is that for a magnetic field strength greater than $\sim 10^{15} \text{ G}$ ($\log \gamma_e \gtrsim 2$), an oscillatory behavior in the adiabatic index begins to appear above a density of $\rho \gtrsim 10^{12} \text{ g cm}^{-3}$. This means that at high densities and fields Γ is significantly affected by the proton fraction even though it is small ($Y_p \lesssim 0.1$). Notice that for $\log \gamma_e \leq 2$, as the density increases above the neutron appearance density ρ_n^* , the oscillatory behavior vanishes since neutron pressure dominates.

This oscillatory behavior has the same physical origin as the well-known de Haas-van Alphen effect (Landau & Lifshitz 1938). It arises as electrons begin to fill the next unoccupied Landau level. Increasing the density increases the occupation of this level and does not lead to a rapidly increased pressure. It would be interesting to understand the physical consequences of the oscillatory behavior of the adiabatic index in strongly magnetized neutron stars. We speculate that this might drive a pulsational instability not unlike classical cepheids. That is, as a region develops low Γ it may be unstable to collapse. At high

density, however, Γ suddenly stiffens and the region bounces as the Γ is restored to a higher value. To explore this possibility, however, it would be necessary to calculate the evolution of the interior of strongly magnetized neutron stars in a realistic magnetohydrodynamical model. This will be the subject of a future study (Suh & Mathews 2000b).

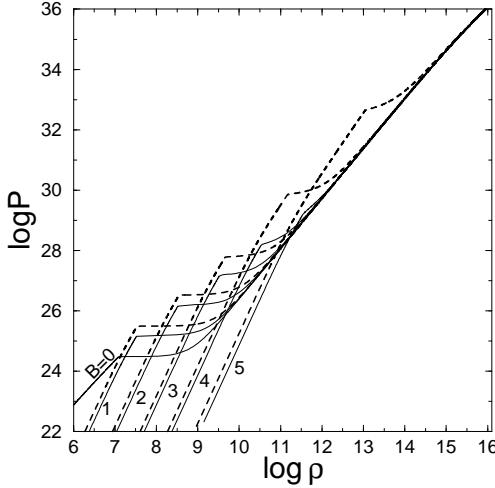


FIG. 4 Pressure P (in dyn/cm^2) vs. total density ρ (in g cm^{-3}) for a npe gas in magnetic fields of various strengths, $\log \gamma_e$. The $B = 0$ line corresponds to the non-magnetic case. The solid line are for the case without the AMM. The dashed lines are for $s_z^n = -1/2$ with the AMM included. Numbers are labeled as in Fig. 3.

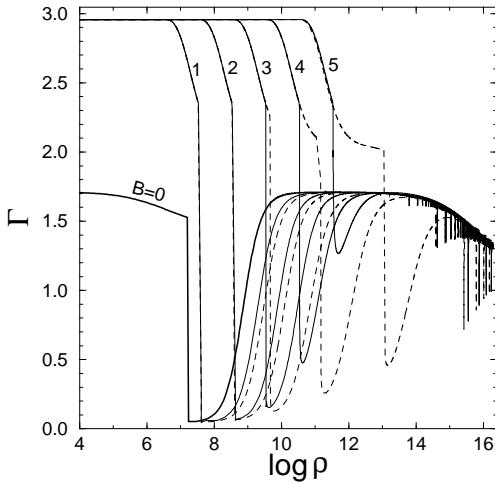


FIG. 5 Adiabatic index Γ vs. ρ (in g cm^{-3}) for a npe gas in various magnetic fields. Curves are labeled as in Fig. 4.

5. MUON PRODUCTION IN A STRONG MAGNETIC FIELD

In order to study the appearance of new particles at high density, let us consider muons in an ideal npe gas. Normally, muons decay to electrons via,

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e. \quad (48)$$

But when the Fermi energy of the electrons approaches the muon rest mass $m_\mu \simeq 105.66 \text{ MeV}$, it becomes energetically favorable for electrons at the top level of Fermi sea to decay into muons with neutrinos and antineutrinos escaping from the star. Hence, above some density, muons and electrons are in equilibrium:

$$\mu^- \leftrightarrow e^-, \quad (49)$$

assuming that the neutrinos leave the star. In this chemical equilibrium, charge conservation implies

$$\mu_\mu = \mu_e. \quad (50)$$

Equilibrium between n , p , and e means

$$\mu_n = \mu_p + \mu_e, \quad (51)$$

and charge neutrality requires

$$n_p = n_e + n_\mu. \quad (52)$$

Now consider the minimum density at which muons are first produced in a strong magnetic field. The threshold condition for muons to appear is given as $n_\mu = 0$. In order to satisfy this condition, ϵ_μ must be unity. If $\epsilon_\mu \neq 1$, the muon number density n_μ is not zero even though the maximum Landau level n_{\max}^μ for muons in a strong magnetic field is zero. Thus, from Eq. (50), we simply obtain

$$\epsilon_e^* = \frac{m_\mu}{m_e}. \quad (53)$$

Then ϵ_p^* and ϵ_n^* are given by chemical equilibrium, Eq. (51), and the charge neutrality condition, Eq. (52), for a given γ_e when $n_\mu = 0$. Actually, since ϵ_p^* and ϵ_n^* depend on the strength of the magnetic field, we solve Eqs. (51) and (52) simultaneously to obtain

$$\epsilon_p^* = \sqrt{\frac{m_p^2 + m_\mu^2 - m_e^2}{m_p^2}} - s_z^n \kappa_p \gamma_p, \quad (54)$$

and

$$\epsilon_n^* = \frac{m_p}{m_n} \epsilon_p^* + \frac{m_e}{m_n} \epsilon_e^*. \quad (55)$$

Thus, the threshold density for the appearance of muons in a magnetic field becomes:

$$\rho_\mu^*(B) = \rho_n(\epsilon_n^*) + \sum_{j=e,p} \frac{\gamma_j}{4\pi^2} \frac{m_j}{\lambda_j^3} \chi_j(\epsilon_j^*). \quad (56)$$

Figure 6 shows the muon threshold density $\rho_\mu^*(B)$ as a function of γ_e . We can see that $\rho_\mu^*(B)$ without the AMM is not affected by magnetic fields less than $B \sim 10^{18} \text{ G}$. But when we consider the AMM, its contribution to $\rho_\mu^*(B)$ becomes important above $B \sim 10^{17} \text{ G}$. However, the equation of state for a npe gas with or without a magnetic field is nearly unaffected by the existence of muons in neutron stars (Canuto & Chiu 1968).

We should also correct for the appearance of hyperons (Bethe & Johnson 1974). Indeed light hyperons with mass less than $\sim 1.2 \text{ GeV}$ are expected to appear at typical neutron star densities depending on the model. But the resulting equation of state is not very different from that of pure neutron matter (Shapiro & Teukolsky 1983).

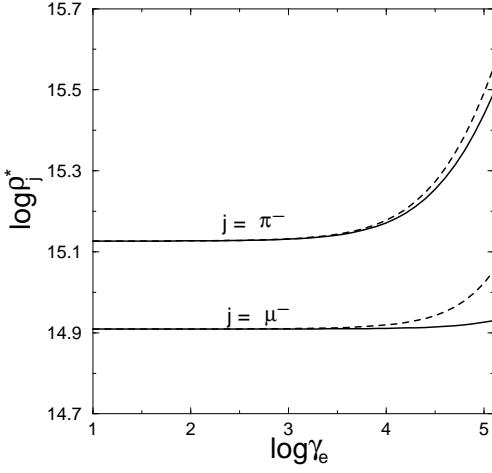


FIG. 6 Appearance density ρ_j^* (in g cm^{-3}) for muons and pions ($j = \mu^-, \pi^-$) as a function of γ_e . The solid line are for the cases without the AMM. The dashed lines are the cases with the AMM included.

6. PION PRODUCTION AND CONDENSATION IN A STRONG MAGNETIC FIELD

At very high density ($\rho \gtrsim \rho_n^*$), neutron-rich nuclear matter is no longer the true ground state of neutron-star matter. It will quickly decay by weak interactions into chemically equilibrated neutron-star matter. Fundamental constituents, besides neutrons, may then include a fraction of protons, hyperons, and possibly more massive baryons. Moreover, a phase transition to quark matter and boson (pion, kaon) condensation are also possible. However, a first-order phase transition by thermal nucleation to quark matter may not occur in a magnetic field (Chakrabarty 1995). Boson condensation in an external magnetic field is also a very subtle problem.

Recently, a true Bose-Einstein condensation (BEC) has been experimentally realized in a system of ^{87}Rb atoms that was confined by magnetic fields and evaporatively cooled (Anderson et al. 1995). An important consequence of the possible appearance of spin zero bosons is that they can form a Bose-Einstein condensation (BEC) at sufficiently low temperatures. An ideal condensation consists of a large number of bosons in a state of zero kinetic energy. This would have at least two implications. One is that the equation of state would be softened, and the other is that the cooling rate from escaping neutrinos would be enhanced (Shapiro & Teukolsky 1983).

Regarding the BEC in a magnetic field, long ago Schafroth (1955) pointed out that for a non-relativistic boson gas, BEC does not take place in the presence of a constant magnetic field. It also was shown by Toms (1994, 1995) that generally a BEC in the presence of a constant magnetic field does not occur in any number of spatial dimensions. However, recently Elm fors et al. (1995) suggested that condensation may occur in three dimension since the Landau ground state can accommodate a large charge density even though it is not exactly a BEC. In particular, Rojas (1996) has shown that a BEC actually may occur in the presence of a constant homogeneous magnetic field in three dimension without requiring the vanishing of the chemical potential. There is, however, no critical temperature at which condensation begins.

Actually, since the criterion for condensation (usually taken as the equality of the chemical potential to the ground state energy) leads to a divergence of the number density, condensation cannot occur in a magnetic field. However, if the chemical potential depends on both temperature and the magnetic field the divergence is avoided and condensation may occur (Rojas 1996). Eventually, in a strong magnetic field and at any non-zero temperature, the number density in the ground state becomes finite. Therefore, in this work we will assume that boson condensation occurs in a strongly magnetized neutron star and that the boson number density has a finite value at sufficiently low temperature.

In order to obtain the energy dispersion relation for a spin zero charged boson, we solve the Klein-Gordon equation in an external magnetic field. Under the same conditions as for charged fermions of Section 2, the dispersion relation for a charged boson in a magnetic field is then given by

$$E_{n_b}^b = [p_z^2 c^2 + m_b^2 c^4 + \hbar c e B n_b]^{1/2}, \quad (57)$$

where $n_b \equiv 2n+1$, ($n = 0, 1, \dots$), n is the principal quantum number of the Landau level, and b denotes charged bosons ($b = \pi^\pm, K^\pm, \dots$). Note that for charged bosons in a magnetic field, their energy state depends on the magnetic field strength even though they are in the lowest Landau level ($n = 0$). Similarly, the boson state density in the absence of a magnetic field,

$$\frac{1}{\hbar^3} \int \frac{d^3 \vec{p}}{(2\pi)^3}, \quad (58)$$

is now replaced with

$$\frac{1}{\hbar^2 c} \sum_{n=0}^{\infty} \int \frac{eB}{(2\pi)^2} dp_z \quad (59)$$

in a magnetic field.

Neutron stars will provide a unique opportunity to verify the hypothesis of boson (pion, kaon) condensation in a strong magnetic field. In this work, as an example, we only consider charged pion condensation via $n \rightarrow p + \pi^-$. Since an ideal cold $npe\mu$ gas allows π^- 's to be produced, let us consider the pion appearance threshold in a strongly magnetized neutron star. If we neglect the strong interaction between pions and nucleons, negatively charged pions are formed through the reaction:

$$n \rightarrow p + \pi^-, \quad (60)$$

in dense neutron matter when the electron chemical potential μ_e exceeds the π^- rest mass, $m_\pi = 139.6$ MeV. Chemical equilibrium requires that the chemical potential should satisfy

$$\mu_n - \mu_p = \mu_e = \mu_\pi \quad \text{and} \quad \mu_e = \mu_\mu \quad (61)$$

Charge neutrality also requires

$$n_e + n_\mu + n_\pi = n_p, \quad (62)$$

that is,

$$\begin{aligned} \sum_{j=e,\mu} \left[\frac{\gamma_j}{2\pi^2 \lambda_j^3} \sum_{n=0}^{n_{\max}^j} \sum_{s_z^j} \sqrt{(\epsilon_j - \kappa_j)^2 - (1 + 2\gamma_j n_f^j)} \right] + n_\pi \\ = \frac{\gamma_p}{2\pi^2 \lambda_p^3} \sum_{n=0}^{n_{\max}^p} \sum_{s_z^p} \sqrt{\epsilon_p^2 - \tilde{m}_p^2}. \end{aligned} \quad (63)$$

Therefore, we can obtain the total mass-energy density of a $npe\mu\pi$ gas:

$$\rho = \sum_{j=n,p,e,\mu} \rho_j + m_\pi n_\pi. \quad (64)$$

For a given ρ , we can determine all the quantities ϵ_e , ϵ_μ , ϵ_p , ϵ_n , and n_π from Eqs. (61) - (64). Figure 7 shows the pion number density n_π as a function of ρ for a given magnetic field. For $\log \gamma_e \leq 3$, the magnetic field effect is negligible.

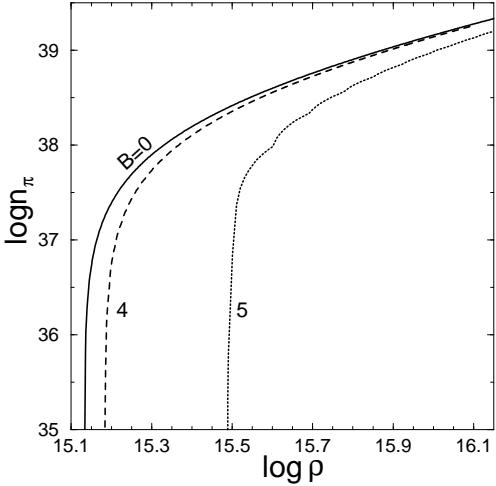


FIG. 7 The pion number density n_π (in cm^{-3}) as a function of ρ (in g cm^{-3}) for the $\log \gamma_e = 4$ (dashed line) and $\log \gamma_e = 5$ (dotted line). The solid line denotes the non-magnetic case ($B=0$).

Similar to the muon appearance, we see that the threshold condition for π^- production is given as $n_\pi = 0$. Finally, the pion production density in a magnetic field is given by

$$\rho_\pi^*(B) = \rho_n(\epsilon_n^*) + \sum_{j=e,p,\mu} \frac{\gamma_j}{4\pi^2} \frac{m_j}{\lambda_j^3} \chi_j(\epsilon_j^*). \quad (65)$$

Figure 6 also shows the pion threshold density $\rho_\pi^*(B)$ as a function of γ_e . We also can see that $\rho_\pi^*(B)$ is not affected by magnetic fields less than $B \sim 10^{17}$ G. However, above this field strength, the magnetic field effect on $\rho_\pi^*(B)$ is important. Note that since pions appear at the extremely high density of $\rho \sim 10^{15}$ g cm^{-3} , the AMM effect is small.

From Eq. (61), the pion momentum is zero ($x_\pi = 0$) because of the fact that in the ground state the condensed pions have zero kinetic energy. This forces the electron chemical potential ϵ_e to remain constant for $\rho > \rho_\pi^*$. In consequence, the electron number density and pressure remain constant as ρ increases. Hence, increasing the pion density contributes to the total mass-energy density but not the pressure. As a result, for a given ρ , the pressure in the condensate phase is always less than in the non-condensate phase. Figure 8 shows the equation of state for an ideal magnetic $npe\mu\pi$ gas with pion condensation. We can see that magnetic fields reduce the pion condensation. However, we still have a distinguishable pion condensate equation of state in strongly magnetized neutron stars. However, at this high density, the AMM does not contribute to the equation of state for the magnetized $npe\mu\pi$ gas although at $\gamma_e \gtrsim 10^5$ the nucleon AMM changes somewhat the pion number density. For

$\log \gamma_e \leq 3$, the magnetic field effect is negligible and nearly the same as the non-magnetic $npe\mu\pi$ gas. Figure 9 shows the adiabatic index Γ as a function of ρ for a pion condensate equation of state. Here we also can see the oscillatory behavior of the adiabatic index. The inclusion of the AMM makes a secondary oscillation in the adiabatic index, but this secondary oscillation disappears gradually as density increases.

7. DISCUSSION

In this work, we have studied the nuclear equation of state for an ideal npe gas in a strong magnetic field. In particular, we have calculated the proton concentration, the threshold densities for neutron, muon, and pion production and pion condensation in a strong magnetic field without and with the effect of the nucleon anomalous magnetic moments. In these calculations, we have shown that the higher Landau levels are significant at high density in spite of the existence of a very strong magnetic field. In particular, at high density, the proton concentration approaches the nonmagnetic limit. As a result, inverse β decay is still suppressed in intense magnetic fields. Therefore, neutron-star rapid cooling is probably not affected by the direct URCA process which is enhanced in strong magnetic fields. In particular, we have obtained the neutron appearance density in a magnetic field when the nucleon AMM is included. We also show that the muon and pion appearance density are not affected by magnetic fields less than about $B \sim 10^{17}$ G. Finally, we here obtained an equation of state with a pion condensate in strong magnetic fields. Magnetic fields reduce the amount of pion condensation. However, we still have distinguishable effects of a pion condensate equation of state in strongly magnetized neutron stars. In addition, we found the oscillatory behavior of the adiabatic index in both strongly magnetized npe and $npe\mu\pi$ gases at high density. Here we speculate that this behavior may also lead to an interior pulsational instability.

It is generally accepted that neutrons and protons in a npe gas are superfluid. The charged pion condensate is also superfluid and superconductive (Migdal et al. 1990). This pion formation and condensation in dense nuclear matter would have the significant consequence (Suh & Mathews 2000c) that the equation of state would be softened. First of all, softening the equation of state reduces the maximum mass of the stars (Baym & Pethick 1975). This softening effect with pion condensation also leads to detectable predictions (Migdal et al. 1990). These are: (i) the enhanced rate of neutron star cooling via neutrinos. (ii) a possible phase transition of neutron stars to a superdense state; and (iii) sudden glitches in pulsar periods. In particular, if the pion condensation occurs in a strong magnetic field, it may significantly affect starquakes.

According to the magnetar model by Duncan and Thompson (1992, 1995), SGRs are caused by starquakes in the outer solid crust of the magnetar. As a colossal magnetic field shifts, it strains the crust with huge magnetic forces and sometimes it cracks. When the crust snaps, it vibrates with seismic waves similar to those of an earthquake. However, in neutron stars they also produce a flash of soft gamma-rays. In addition, Cheng and Dai (1998) recently suggested that SGRs may be rapidly rotating magnetized strange stars with superconducting cores.

Although such models can explain some crucial features, there are still several unsettled issues (Liang 1995). Therefore, superconducting cores with a charged boson (pion and/or kaon) condensate in magnetars might be an alternative model to explain the energy source of soft gamma-rays from magnetars.

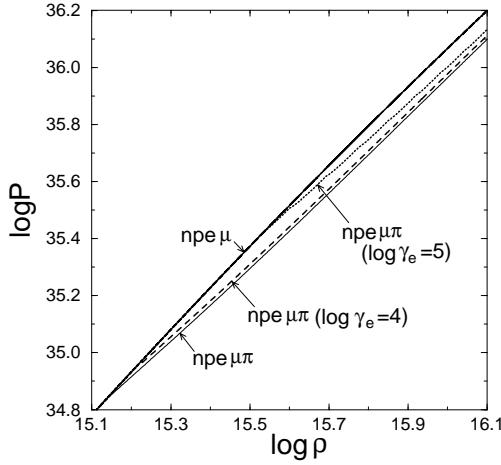


FIG. 8 The equation of state for an ideal magnetic $npe\mu\pi$ gas with pion condensation. The thick and thin solid lines are non-magnetic cases ($B = 0$). The dashed and dotted lines are magnetic cases for $\log\gamma_e = 4$ and 5, respectively.

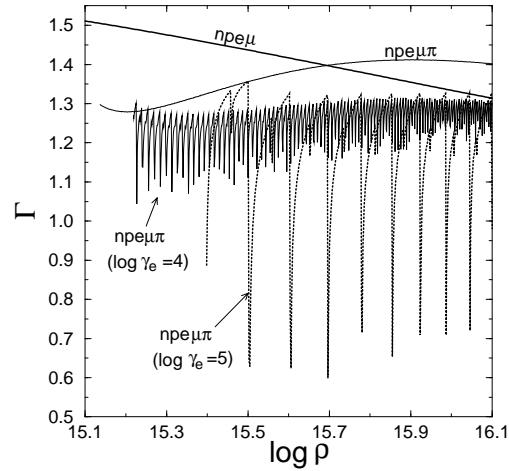


FIG. 9 Adiabatic index Γ as a function of ρ (in g cm^{-3}) for a pion condensate equation of state for magnetic field strengths $\log\gamma_e = 4$ and 5. The thick and thin solid lines are for non-magnetic ($B = 0$) $npe\mu$ and $npe\mu\pi$ cases.

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